# THE BIG BANG QUANTUM COSMOLOGY: THE MATTER-ENERGY PRODUCTION EPOCH

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**Abstract**

The exactly solvable quantum model of the homogeneous, isotropic and closed universe in the matter-energy production epoch is considered. It is assumed that the universe is originally filled with a uniform scalar field and a perfect fluid which defines a reference frame. The stationary state spectrum and the wave functions of the quantum universe are calculated. In this model the matter-energy in the universe has a component in the form of a condensate of massive zero-momentum excitation quanta of oscillations of primordial scalar field. The mean value of the scale factor of the universe in a given state is connected with the mass of a condensate by a linear relation. The nucleation rate of the universe from the initial cosmological singularity point is calculated. It is demonstrated that the process of nucleation of the universe can have an exponential (explosive) nature. The evolution of the universe is described as transitions with non-zero probabilities between the states of the universe with different masses of a condensate.

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## 1. Introduction

The method of constraint system quantization can be taken as a basis of quantum theory of gravity suitable for the investigation of cosmological systems [1]. As is well known the structure of constraints in general relativity is such that variables which correspond to true dynamical degrees of freedom cannot be singled out from canonical variables of geometrodynamics. The reason behind this difficulty is the absence of predetermined way of identifying spacetime events in generally covariant theory [2].

In contrast to gravitational field in a void, the consideration of gravitational field coupled with matter allows to use matter in order to give an invariant meaning to space-time points [2, 3, 4]. Using material reference systems, one can address conceptual problems of not only classical, but also quantum gravity [5].

In Ref. [2] a scheme to include reference frames in general relativity by means of an introduction of coordinate conditions was developed. A task at this point is to find a material source in the Einstein equations which determines a reference frame and has no unphysical properties. This approach was applied in Refs. [6, 7] in order to solve the problems of quantum theory of gravity in the minisuperspace model with a material source in the form of relativistic matter (radiation) which defines the reference frame. The variables which describe radiation mark spacetime events, since the reference frame is considered as a dynamical system. These variables play the role of the canonical coordinates which determine an embedding in the encompassing spacetime. At the same time the new constraints turn out to be linear with respect to the momenta canonically conjugate with them. Such an approach allows to obtain the time-dependent Schrödinger equation and to define a conserved positive definite inner product.

On the other hand it is well known that during the quantization of different model systems in gravity one can use a perfect fluid as a reference frame [3, 8, 9]. In this case one deals directly with a physical medium without coordinate conditions as an intermediate under the construction of a material source. This leaves aside problems connected with the necessity to ensure that the action is coordinate invariant and that a material source which determines a reference frame has correct physical properties. Relativistic matter is a special case of a perfect fluid and as the simplest physical system can be used to define a reference frame.

In Section 2 the summary of ideas that lead to the fundamental equations of quantum model of the universe is given.

In Section 3 the stationary state spectrum and the wave functions of the quantum universe filled with primordial matter in the form of a uniform scalar field and a perfect fluid (radiation) which defines a reference frame are calculated. It is shown that the matter-energy in the universe has a component in the form of a condensate of massive zero-momentum excitation quanta of oscillations of a primordial scalar field. The mean value of the scale factor in a given state of the universe is calculated. This mean value is determined by the mass of a condensate and the mean deviation from an equilibrium state that can be neglected in the semi-classical limit. It is shown that the universe with the Planck mass of a condensate in the ground (vacuum) state has the mean value of the scale factor that practically coincides with the Planck length.

In Section 4 the nucleation rate of the universe from the initial cosmological singularity point is calculated and it is demonstrated that the process of nucleation of the universe can have an exponential (explosive) nature. This phenomenon can be identified with the initial moment of the Big Bang. It is shown that the greater masses of a condensate of the nucleating universe correspond to its larger initial sizes.

In Section 5 the probabilities of transitions between the states of the universe with different masses of a condensate are calculated. It is shown that the probability

of transition from the vacuum state of the universe to another state obeys the Poisson distribution.

In Section 6 a conclusion is given.

## 2. Constraint system quantization in the presence of a medium which defines a reference frame

Let us consider a cosmological system (universe) with the action

$$S = S_{E-H} + S_M + S_{PF}, \tag{1}$$

where

$$S_{E-H} = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-g}\, R \tag{2}$$

is the Einstein-Hilbert action for gravitational field, $S_M$ is the action of matter fields,

$$S_{PF} = \frac{1}{c} \int d^4x \sqrt{-g}\, \{ -\rho(\rho_0, s) \,+\, \lambda [g^{\mu\nu} U_\mu U_\nu - 1] \\ + \rho_0 U^\nu [s\, \Theta_{,\,\nu} \,-\, \widetilde{\lambda}_{,\,\nu} \,+\, \beta_i\, \alpha^i_{\,,\,\nu}]\, \} \tag{3}$$

is the action of a perfect fluid (macroscopic bodies) [10, 11, 12, 13, 14], which defines material reference frame, where $\rho$ is the energy density as a function of the density of the rest mass $\rho_0$ and the specific entropy $s$, $U^\nu$ is the four-velocity, $\lambda$ is a Lagrange multiplier that ensures normalization of $U^\nu$. The $\Theta$, $\widetilde{\lambda}$, $\beta_i$, $\alpha^i$ are scalar fields. Here $\Theta$ has a meaning of the thermasy or the “potential” for the temperature $T$, $T = \Theta_{,\nu} U^\nu$. The $\widetilde{\lambda}$ is the “potential” for the specific free energy $f$ taken with an inverse sign, $f = -\,\widetilde{\lambda}_{,\nu} U^\nu$. The $\beta_i$ and $\alpha^i$ are Lagrange multipliers and Lagrangian coordinates for a fluid on a spacelike hypersurface respectively (one should introduce them into the action in order to describe rotational flows and, generally speaking, an incorporation of one pair of the variables $\beta_i$ and $\alpha^i$ into the action would be enough, but then their direct physical interpretation will be lost). The thermodynamic variables are related via the first law of thermodynamics

$$d\rho = h\, d\rho_0 \,+\, \rho_0\, T\, ds, \tag{4}$$

where $h = \frac{\rho + p}{\rho_0}$ is the specific enthalpy which plays the role of inertial mass, $p$ is the pressure.

The components of the metric tensor $g^{\mu\nu}$, the matter fields contained in $S_M$, and the values $\rho_0$, $s$, $U^\nu$, $\Theta$, $\lambda$, $\widetilde{\lambda}$, $\beta_i$ and $\alpha^i$ play the role of generalized variables. All equations of classical theory of gravity follow from the principle of least action

$$\delta S = 0, \tag{5}$$

where all independent variables are varied [14].

Let us assume that the universe is homogeneous, isotropic, closed and described by the Robertson-Walker metric

$$ds^2 = N^2(t)c^2dt^2 - a^2(t)d\Omega^2, \tag{6}$$

where $a(t)$ is the cosmic scale factor, $N(t)$ is the lapse function that specifies the time reference scale, $t$ is the time variable, $d\Omega^2$ is an interval element on a unit three-sphere. We choose the uniform scalar field $\phi$ with the potential energy density (potential) $V(\phi)$ as a matter. The choice of such a field as a primordial matter seems to be reasonable, since any other fields (vector or spinor, for instance), being non-averaged over all space variables at every instant of time, can destroy the supposed property of homogeneity and isotropy of the universe.

In the model of the universe under consideration the action (1) can be reduced to the form [14]

$$S = \int dt \left( \pi_a \frac{da}{dt} + \pi_\phi \frac{d\phi}{dt} + \pi_\Theta \frac{d\Theta}{dt} + \pi_{\widetilde{\lambda}} \frac{d\widetilde{\lambda}}{dt} - H \right), \tag{7}$$

where $\pi_a$, $\pi_\phi$, $\pi_\Theta$, $\pi_{\widetilde{\lambda}}$ are the momenta canonically conjugate with the variables $a$, $\phi$, $\Theta$, $\widetilde{\lambda}$, and it is taken into account that the momenta conjugate with the variables $\rho_0$, $s$ and $N$, vanish identically. In this model $U^0 = \frac{1}{N}$, $U^i = 0$, the condition $g^{\mu\nu}U_\mu U_\nu = 1$ is contained explicitly in the variational principle and, therefore, the term with $\lambda$ should be dropped. Since the variables $\beta_i$ and $\alpha^i$ describe rotational flows which would create a preferential direction in space, the terms with $\beta_i$ and $\alpha^i$ should be dropped as well and one should consider irrotational (potential) flows of a perfect fluid only.

The $H$ in Eq. (7) is the Hamiltonian

$$\begin{aligned} H = N \left( -\frac{3\pi c^4}{4G} \right) \frac{1}{a} \left\{ \left( \frac{2G}{3\pi c^3} \right)^2 \pi_a^2 + a^2 - \frac{G}{3\pi^3 c^2} \frac{\pi_\phi^2}{a^2} - \frac{8\pi G}{3c^4} a^4 [\rho + V(\phi)] \right\} \\ + \lambda_1 \left\{ \pi_\Theta - \frac{1}{c} 2\pi^2 a^3 \rho_0 \, s \right\} + \lambda_2 \left\{ \pi_{\widetilde{\lambda}} + \frac{1}{c} 2\pi^2 a^3 \rho_0 \right\}, \end{aligned} \tag{8}$$

where $\lambda_1$ and $\lambda_2$ are Lagrange multipliers. The function $N$ in (8) plays also the role of a Lagrange multiplier (like in ADM-formalism [15]). The variation of the action (7) with respect to $N$, $\lambda_1$ and $\lambda_2$ leads to three constraint equations.

Taking into account the conservation laws (see below) and vanishing of the momenta conjugate with $\rho_0$ and $s$, one can discard degrees of freedom corresponding to these variables and convert the second-class constraints into first-class constraints in accordance with Dirac's proposal [1]. In quantum theory first-class constraint equations become constraints on the wave function $\Psi$.

Replacing the momenta by the operators

$$\begin{aligned} \pi_a \to \hat{\pi}_a &= -i\hbar\partial_a, \quad \pi_\phi \to \hat{\pi}_\phi = -i\hbar\partial_\phi, \\ \pi_\Theta \to \hat{\pi}_\Theta &= -i\hbar\partial_\Theta, \quad \pi_{\widetilde{\lambda}} \to \hat{\pi}_{\widetilde{\lambda}} = -i\hbar\partial_{\widetilde{\lambda}}, \end{aligned}$$

which satisfy the commutation relations

$$[a, \hat{\pi}_a] = i\,\hbar, \quad [\phi, \hat{\pi}_\phi] = i\,\hbar, \quad [\Theta, \hat{\pi}_\Theta] = i\,\hbar, \quad [\widetilde{\lambda}, \hat{\pi}_{\widetilde{\lambda}}] = i\,\hbar,$$

while all other commutators vanish, we obtain

$$\left\{ -\left(\frac{2G\hbar}{3\pi c^3}\right)^2 \partial_a^2 + a^2 + \frac{G\hbar^2}{3\pi^3 c^2}\,\frac{1}{a^2}\,\partial_\phi^2 - \frac{8\pi G}{3c^4}\,a^4[\rho + V(\phi)] \right\} \Psi = 0, \tag{9}$$

$$\left\{ -i\,\hbar\,\partial_\Theta - \frac{1}{c}\,2\pi^2 a^3 \rho_0\, s \right\} \Psi = 0, \tag{10}$$

$$\left\{ -i\,\hbar\,\partial_{\widetilde{\lambda}} + \frac{1}{c}\,2\pi^2 a^3 \rho_0 \right\} \Psi = 0. \tag{11}$$

In Eq. (9) the factor ordering parameter associated with a possible ambiguity in the choice of an explicit form of the operator $\hat{\pi}_a^2$ is assumed to be zero.

It is convenient to pass from the generalized variables $\Theta$ and $\widetilde{\lambda}$ to the non-coordinate co-frame

$$\begin{aligned} h\,d\tau &= s\,d\Theta - d\widetilde{\lambda}, \\ h\,dy &= s\,d\Theta + d\widetilde{\lambda}, \end{aligned} \tag{12}$$

where $\tau$ is proper time in every point of space. It is easy to prove that the corresponding derivatives commute between themselves,

$$[\partial_\tau,\, \partial_y] = 0.$$

Then Eqs. (10) and (11) reduce to the form

$$\left\{ -i\,\hbar\,\partial_{\tau_c} - \frac{1}{c}\,2\pi^2 a^3 \rho_0 \right\} \Psi = 0, \qquad \partial_y \Psi = 0, \tag{13}$$

where $d\tau_c = h\,d\tau$.

From the variation of action with respect to $\widetilde{\lambda}$ it follows the conservation law

$$a^3 \rho_0 = \text{const}. \tag{14}$$

It describes a conserved macroscopic value which characterizes the number of particles. For example, if a perfect fluid is composed of baryons, then Eq. (14) describes the conservation of baryon number.

From the variation of action with respect to $\Theta$ it follows that the specific entropy is conserved

$$s = \text{const}.$$

From Eq. (9) one can see that it is convenient to take the matter component $\rho$ in the form of relativistic matter (radiation) with the equation of state $p = \frac{1}{3}\rho$. In this case

$$a^4 \rho = \text{const.} \tag{15}$$

Denoting the constant in Eq. (15) as $E$, while in Eq. (14) as $E_0$, we obtain the equations

$$\left\{ - i\, \hbar\, \partial_{\tau_c} - \frac{1}{c}\, 2\pi^2 E_0 \right\} \Psi = 0, \tag{16}$$

$$\left\{ - \left( \frac{2G\hbar}{3\pi c^3} \right)^2 \partial_a^2 + a^2 + \frac{G\hbar^2}{3\pi^3 c^2}\, \frac{1}{a^2}\, \partial_\phi^2 - \frac{8\pi G}{3c^4}\, [a^4 V(\phi) + E] \right\} \Psi = 0, \tag{17}$$

where, according to Eq. (13), the wave function $\Psi$ does not depend on $y$.

Eq. (16) describes the evolution of the state $\Psi$ with respect to the time variable $\tau_c$. Eq. (17) does not contain $\tau_c$ explicitly. At this point there is a close analogy with properties of closed systems in quantum mechanics.

The constants $E_0$ and $E$ are dimensional quantities, $[E_0]$ = energy, $[E]$ = energy $\times$ length. It is convenient to rewrite Eqs. (16) and (17) for dimensionless quantities. With that end in view we bring in correspondence

$$a \to \frac{a}{l_P}, \quad \phi \to \frac{\phi}{\phi_P}, \quad \tau_c \to \frac{\tau_c}{l_P}, \quad V \to \frac{V}{\rho_P}, \quad E_0 \to \frac{4\pi^2}{m_P c^2}\, E_0, \quad E \to \frac{4\pi^2}{\hbar c}\, E,$$

where we have dimensionless quantities from the left, and

$$l_P = \sqrt{\frac{2G\hbar}{3\pi c^3}}, \quad \phi_P = \sqrt{\frac{3c^4}{8\pi G}}, \quad t_P = \frac{l_P}{c}, \quad \rho_P = \frac{3c^4}{8\pi G l_P^2}, \quad m_P = \frac{\hbar}{l_P c}$$

are the Planck values of length, scalar field, time, energy density and mass, respectively. Then Eqs. (16) and (17) in new dimensionless variables take the form

$$\left\{ - i\, \partial_{\tau_c} - \frac{1}{2}\, E_0 \right\} \Psi = 0, \tag{18}$$

$$\left\{ - \partial_a^2 + \frac{2}{a^2}\, \partial_\phi^2 + a^2 - a^4 V(\phi) - E \right\} \Psi = 0. \tag{19}$$

Eq. (18) has a particular solution in the form

$$\Psi = \mathrm{e}^{\frac{i}{2} E_0 \tau_c} \psi, \tag{20}$$

where $\psi$ is a function which depends on $a$ and $\phi$ only and is determined by Eq. (19). If we pass from the time variable $\tau_c$ to $\overline{\tau} = \frac{E_0}{2E}\, \tau_c$, as a result we arrive in Eqs. (19) and (20) to an analogy with the Schrödinger equation for stationary states. We have obtained Eqs. (19) and (20) previously in Refs. [6, 7] within the bounds of

the scheme for incorporating of a reference system in general relativity through the introduction of a coordinate condition.

Let us note that we can obtain Eqs. (18) and (19) even without an introduction of proper time by means of Eqs. (12). It is possible to build a time variable from the matter variables (e.g. Refs. [8, 9]). We can consider e.g. $\Theta$ as a time variable [14]. (On the correspondence between the thermasy and proper time see Ref. [16].)

Second order partial differential equation (19) is given on the intervals $0 \leq a < \infty$ and $-\infty < \phi < \infty$. It should be supplemented with boundary conditions. We suppose that at $a \to 0$ the function $\psi(a, \phi) \to$ const. At $a \to \infty$ its form depends on the properties of the potential $V(\phi)$. It will determine the behaviour of $\psi$ on the boundaries $\phi \to \pm\infty$.

As is well known, the energy density of a uniform scalar field can be written as

$$\rho_\phi = \frac{2}{a^6}\,\pi_\phi^2 + V(\phi), \tag{21}$$

where we have used an explicit form of the momentum $\pi_\phi$ canonically conjugate with $\phi$ [7],

$$\pi_\phi = \frac{1}{2}\,a^3\,\frac{d\phi}{d\tau}. \tag{22}$$

By analogy we determine the energy density operator for the field $\phi$,

$$\hat{\rho}_\phi = -\,\frac{2}{a^6}\,\partial_\phi^2 + V(\phi). \tag{23}$$

Then the operator of the total energy density in the quantum system under consideration will have the form

$$\hat{\rho}_{tot} = \hat{\rho}_\phi + \rho, \tag{24}$$

where $\rho$ is the same as in Eq. (15), and Eq. (19) after multiplying from the left by $a^{-4}$ and averaging over the state $\Psi$ normalized in one way or another (see below), can be written as

$$\langle \hat{G}_{00} \rangle = 3 \langle \hat{T}_{00} \rangle, \tag{25}$$

where the operators are

$$\hat{G}_{00} = \frac{3}{a^4}\,\left(\hat{\pi}_a^2 + a^2\right), \qquad \hat{T}_{00} = \hat{\rho}_{tot}. \tag{26}$$

Comparing Eqs. (25) and (26) with the Einstein equations in general relativity we arrive at a conclusion that the operator $\hat{G}_{00}$ can be considered as a generalization to quantum theory of the correspondent component of the Einstein tensor, while $\hat{T}_{00}$ is the operator of (00)-component of the energy-momentum tensor (stress tensor) of matter. The relation (25) can be used to find quantum corrections to the Einstein-Friedmann equation of classical theory of gravity.

## 3. Stationary states of the quantum universe

Let us study the properties of the quantum universe described by the steady-state equation (19). Since the universe is supposed to be closed, then one can introduce a notion of the mass of the universe as a product of its matter density and a comoving volume. In the units under consideration the mass of a scalar field in the universe with the scale factor $a$ is equal to

$$M_\phi = \frac{1}{2}\, a^3 \rho_\phi. \tag{27}$$

This value will be associated with the operator

$$\hat{H}_\phi = \frac{1}{2}\, a^3\, \hat{\rho}_\phi, \tag{28}$$

where $\hat{\rho}_\phi$ is (23), while $\frac{1}{2}\, a^3$ is a comoving volume. Then Eq. (19) takes the form

$$\left( - \partial_a^2 + a^2 - 2a\hat{H}_\phi - E \right) \psi = 0. \tag{29}$$

Further let us suppose that the potential $V(\phi)$ is a smooth function of $\phi$. Let there exists a value of the field $\phi = \sigma$ at which the function $V(\phi)$ has a minimum, while the value $\sigma$ itself corresponds to the true vacuum of the field $\phi$, $V(\sigma) = 0$ (an absolute minimum [17]). Then near the point $\phi = \sigma$ the following representation is valid

$$V(\phi) = \frac{m_\sigma^2}{2}\, (\phi - \sigma)^2, \tag{30}$$

where $m_\sigma^2 = [d^2 V(\phi)/d\phi^2]_\sigma > 0$.

Let us make a scaling transformation of the field $\phi$ and introduce a new variable $x$ which describes a deviation of the field $\phi$ from its vacuum state $\sigma$,

$$x = \left( \frac{m_\sigma a^3}{2} \right)^{1/2} (\phi - \sigma). \tag{31}$$

The operator (28) takes the form

$$\hat{H}_\phi = \frac{m_\sigma}{2} \left( - \partial_x^2 + x^2 \right). \tag{32}$$

Let us introduce the eigenfunctions of harmonic oscillator $u_k(x)$ as a solution of the equation

$$\left( - \partial_x^2 + x^2 \right)\, u_k(x) = (2k + 1)\, u_k(x), \tag{33}$$

where $k = 0,\, 1,\, 2,\, ...$ is the number of the state of the oscillator,

$$u_k(x) = [\sqrt{\pi} k!\, 2^k]^{-1/2}\, \mathrm{e}^{-x^2/2} H_k(x),$$

$H_k(x)$ are the Hermitian polynomials. Then we find

$$\hat{H}_\phi u_k = M_k u_k, \tag{34}$$

where

$$M_k = m_\sigma \left( k + \frac{1}{2} \right). \tag{35}$$

The value $M_k$ can be interpreted as an amount of matter-energy (or mass) in the universe related to a scalar field. In the second quantization formalism this energy is represented in the form of a sum of excitation quanta of the spatially coherent oscillations of the field $\phi$ about the equilibrium state $\sigma$, $k$ is the number of these excitation quanta. Such oscillations correspond to a condensate of zero-momentum $\phi$ quanta with the mass $m_\sigma$. The mass $m_\sigma$ is determined by the curvature of the potential $V(\phi)$ near $\phi = \sigma$.

Taking into account (34) we shall look for the solution of Eq. (29) in the form of the superposition of the states with all possible values of the quantum number $k$ (and, correspondingly, with all possible masses $M_k$)

$$\psi = \sum_k f_k(a)\, u_k(x). \tag{36}$$

Substituting (36) into (29) and using the orthonormality condition for the states $u_k(x)$, $\langle u_k | u_{k'} \rangle = \delta_{kk'}$, we obtain the equation for $f_k(a)$,

$$\left( - \partial_a^2 + a^2 - 2aM_k - E \right) f_k(a) = 0. \tag{37}$$

This equation has an analytical solution decreasing at $a \to \infty$ which has the form of the wave function of an oscillator perturbed by the mass term $-2aM_k$,

$$f_k(a) \equiv f_{n,k}(a) = N_{n,k}\, \mathrm{e}^{-\frac{1}{2}(a-M_k)^2} H_n(a - M_k) \tag{38}$$

at

$$E \equiv E_{n,k} = 2n + 1 - M_k^2, \tag{39}$$

where $n = 0,\, 1,\, 2, ...$ is the number of the state of the quantum universe at a given $k$-state of a condensate (with the mass $M_k$) in the potential well

$$U(a) = a^2 - 2aM_k. \tag{40}$$

For the states $f_{n,k}(a)$, normalized by the condition $\langle f_{n,k} | f_{n,k} \rangle = 1$, the normalization factor $N_{n,k}$ is equal to

$$\begin{aligned} N_{n,k} \quad = \quad & \Big\{ 2^{n-1} n! \sqrt{\pi}\, [\operatorname{erf} M_k + 1] \\ & - \mathrm{e}^{-M_k^2} \sum_{l=0}^{n-1} \frac{2^l n!}{(n-l)!}\, H_{n-l}(M_k)\, H_{n-l-1}(M_k) \Big\}^{-\frac{1}{2}}, \end{aligned} \tag{41}$$

where

$$\operatorname{erf} M = \frac{2}{\sqrt{\pi}} \int_0^M dt\, \mathrm{e}^{-t^2}$$

is a probability integral. From the properties of the function $\operatorname{erf} M$ and the properties of the Hermitian polynomials $H_n(M)$ it follows that the normalization factor (41) for $M_k > 1$ is equal to

$$N_{n,k} = \left\{2^n n! \sqrt{\pi} - O\left((2M_k)^{2n-1}\, \mathrm{e}^{-M_k^2}\right)\right\}^{-\frac{1}{2}}. \qquad (42)$$

One can neglect the exponential addition for the states of the quantum universe with the large enough masses $M_k$ of a condensate. In this case $N_{n,k}$ does not depend on $k$, and its numerical value coincides with the numerical value of the normalization factor of the wave function of ordinary harmonic oscillator in the state $n$.

According to (38) and (39) the quantum states of the universe are characterized by two quantum numbers $n$ and $k$. The mean value of the scale factor $a$ in the state (38),

$$\bar{a} = \langle f_{n,k} | a | f_{n,k} \rangle, \qquad (43)$$

equals to

$$\bar{a} = M_k + \bar{\xi}, \qquad (44)$$

where

$$\bar{\xi} = N_{n,k}^2\, 2^{n-1} n!\, \mathrm{e}^{-M_k^2} \left\{1 + \sum_{l=0}^{n-1} \frac{2^{l-n}}{(n-l)!}\, H_{n-l}(M_k)\, H_{n-l-1}(M_k)\right\}$$
$$= \frac{N_{n,k}^2}{2}\, n!\, \mathrm{e}^{-M_k^2} \left\{2^n + O\left((2M_k)^{2n-1}\right)\right\}. \qquad (45)$$

According to (44) and (45) the universe with the Planck mass of a condensate, $M_k = 1$, in the ground (vacuum) state, $n = 0$, is characterized by the mean value $\bar{a}_{n=0} = 1.11$ which coincides with the Planck length by an order of magnitude. For the states with $M_k > 1$ the mean value (44) does not depend on $n$ to within a small summand $\sim O((2M_k)^{2n-1}\, \mathrm{e}^{-M_k^2})$ and is determined by the mass $M_k$ only,

$$\bar{a} = M_k \qquad \text{at} \qquad M_k \gg 1. \qquad (46)$$

The mass $M_k$ determines also the value of an absolute minimum of the effective potential $U(a)$ (40) of Eq. (37), $U(M_k) = -\,M_k^2$.

## 4. The nucleation rate of the universe from the initial cosmological singularity point

The wave function $f_{n,k}(a)$ describes the universe in the state with the quantum numbers $n$ and $k$ and depends on the scale factor $a$. The ground state of the quantum

universe is characterized by the Planck parameters. The wave function $f_{n,k}(a)$ itself can be considered at the point $a = 0$. According to quantum field theory a particle decay rate is determined by the expression

$$\Gamma_\psi = \overline{v\,\sigma_r}\,|\psi(0)|^2, \tag{47}$$

where $v$ is the relative velocity of decay products, $\sigma_r$ is the reaction cross-section, and a bar means an averaging over non-recording parameters (e.g., over initial spin states), $\psi(0)$ is the wave function of a particle before the decay in the origin (at zero distance). In accordance with that the value

$$\Gamma_{n,k} = \overline{v\,\sigma_r}\,|f_{n,k}(0)|^2, \tag{48}$$

can be interpreted as a rate of nucleation of the universe in the $n, k$-state from the initial cosmological singularity point $a = 0$. In accordance with classical view the nucleation of the universe is the process in time with an expansion initiation at some instant $\tau$ which it is convenient to choose as $\tau = 0$. The velocity $v$ can be naturally identified with the expansion rate $v = \frac{da}{d\tau}$ at $\tau = 0$, where $\tau$ is some time parameter which ensures the boundary condition $a(\tau = 0) = 0$. The cross-section $\sigma_r$ will be set equal to $\sigma_r = \pi\,a^2$. The dependence $a(\tau)$ can be found from the condition of finiteness of $\overline{v\,\sigma_r}$ at the point $a = 0$. We put

$$\overline{v\,\sigma_r} \equiv \lim_{a\to 0}\left(\frac{da}{d\tau}\,\pi\,a^2\right) = \text{const.} \tag{49}$$

It is easy to make sure that at const $\neq 0$ the single possible case is $a(\tau) \sim \tau^{1/3}$ which is realized when primordial matter is described by the extremely rigid equation of state at the point $a(\tau = 0) = 0$,

$$p_{in} = \rho_{in},$$

where $p_{in}$ and $\rho_{in}$ are the pressure and energy density of a primordial scalar field at the moment of nucleation of the universe. From the point of view of semi-classical approximation such an equation of state is realized in the primordial scalar field $\phi$ taken in the state of its true vacuum $\phi = \sigma$, where $V(\sigma) = 0$,

$$\rho_\sigma = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2_\sigma = p_\sigma, \tag{50}$$

when all energy of the field $\phi$ is concentrated in its kinetic part. Such a state is unstable and should turn into the state with a condensate of $\phi$ quanta. This transition will look like a nucleation of the universe from the point $a = 0$ with the extremely rigid equation of state (50) and the wave function $f_{n,k}(0)$, the square of which determines the rate of such a process in accordance with (48). (Physics of transition from $a = 0$ to $\bar{a} \neq 0$ is considered below at the end of this section.)

Using an explicit form of the function $f_{n,k}(a)$ (38) and keeping the main term in $N_{n,k}$ (42) only, we find

$$\Gamma_{n,k} \simeq \text{const}\, \frac{2^n}{\sqrt{\pi}}\, P(n), \tag{51}$$

where

$$P(n) = \frac{\langle n \rangle^n}{n!}\, \mathrm{e}^{-\langle n \rangle}$$

is the Poisson distribution with the mean value $\langle n \rangle = M_k^2$ of the quantum number $n$. The total nucleation rate of the universe $\Gamma = \sum_n \Gamma_{n,k}$ is given by the mean value $\langle 2^n \rangle$ over the Poisson distribution. It appears to be exponentially high

$$\Gamma \simeq \frac{\text{const}}{\sqrt{\pi}} \exp\{M_k^2\}. \tag{52}$$

According to quantum theory the square of modulus of the wave function at the origin determines the particle number density at this point. In the case of the quantum universe

$$|f_{n,k}(0)|^2 \sim L_{n,k}^{-3}, \tag{53}$$

where $L_{n,k}$ is the linear dimension of the region from which the universe nucleates. Hence we find that

$$L_{n,k} \sim \left( \frac{\sqrt{\pi} n!}{2^n M_k^2} \right)^{1/3} \exp\{\frac{M_k^2}{3}\}. \tag{54}$$

For $n = 0$ and $M_k = 1$ we have $L_{0,k} \sim 1.69$. This value is consistent with the mean value $\bar{a}_{n=0} = 1.11$ calculated above for the universe with the Planck mass of a condensate.

Given estimations show that the universe nucleates with a finite rate into the state with the Planck parameters (mass and spatial dimensions). The larger masses of a condensate of nucleating universe correspond to the larger primordial dimensions. The total nucleation rate of the universe obeys an exponential law and corresponds to an "explosion" from the point $a = 0$. It can be identifies with zero time of the Big Bang.

As we have shown in Ref. [18] a condensate of $\phi$ quanta with the total mass $M_k \neq 0$ has an antigravitating property and is described by the vacuum equation of state $p_k = -\rho_k$, where $\rho_k = \frac{2M_k}{\bar{a}^3}$. The universe from an unstable state with the extremely rigid equation of state at the point $a = 0$ passes into the ground state with the Planck mass of a condensate and the Planck scale factor. As soon as the mass of a condensate reaches nonzero values the equation of state of a primordial scalar field changes from the extremely rigid equation of state to the vacuum one and a condensate acquires an antigravitating property. The growth of $M_k$ leads to the growth of antigravitation and as a consequence triggers a subsequent growth of $\bar{a}$ of the quantum universe which at that undergoes an accelerating expansion.

## 5. Probabilities of transitions between the states of the universe with different masses of a condensate

According to (36), (38) the function $f_{n,k}(a)$ can be interpreted as a state vector which describes the universe in the $n$-th state with the mass of a condensate $M_k$. The states $f_{n,k}$ are orthogonal with respect to the quantum number $n$ up to small terms $\sim \exp\{-M_k^2\}$. If the states $f_{n,k}$ and $f_{n',k'}$ correspond to different masses, $M_k \neq M_{k'}$, then they are eigenfunctions of different operators and therefore, generally speaking, are nonorthogonal between themselves. The correspondent overlap integral is $\langle f_{n,k}|f_{n',k'}\rangle \neq 0$. The evolution of the universe takes place is such a way that it passes from, say, the $n,k$-state in one potential well $U(a)$ (40) into the state with the quantum numbers $n', k'$ in another well, where the number of the state $n'$ may differ from $n$ or be equal to it, but an index $k'$ that numbers a quantity of $\phi$ quanta differs from $k$.

Eq. (37) describes the stationary states. Therefore when calculating the transition probability $w(n,k \rightarrow n',k')$ we use the model of the instantaneous change of the state of quantum system. Then

$$w(n,k \rightarrow n',k') = |\langle f_{n',k'}|f_{n,k}\rangle|^2. \tag{55}$$

It is easy to show that the normalization condition

$$\sum_{n'} w(n,k \rightarrow n',k') = 1 \tag{56}$$

is satisfied to within exponentially small terms.

Using the explicit form of the function $f_{n,k}(a)$ (38), integrating by parts, and neglecting the terms $\sim \exp\{-\,M_k^2\}$ for $M_k \gg 1$ we find

$$\langle f_{n',k'}|f_{n,k}\rangle = 2\sqrt{\pi}\, N_{n',k'} N_{n,k}\, \xi_0^{n-n'}\, \mathrm{e}^{-\,\xi_0^2} \sum_{i=0}^{n'} (-1)^i \, \frac{2^{n'-i}\, n'!\, n!\, \xi_0^{2i}}{i!(n'-i)!(n-n'+i)!} \tag{57}$$

for $n \neq 0$ and

$$\langle f_{n',k'}|f_{0,k}\rangle = 2\sqrt{\pi}\, N_{n',k'} N_{0,k}\, \xi_0^{n'}\, \mathrm{e}^{-\,\frac{1}{4}\xi_0^2}, \tag{58}$$

where we denote $\xi_0 = M_{k'} - M_k$. The transition probability (55) at $n' > n \geq 1$ is equal to

$$w(n,k \rightarrow n',k') \simeq \frac{1}{2^{n'-n-2}}\, \frac{n'!}{n![(n'-n)!]^2}\, \xi_0^{2(n'-n)}\, \mathrm{e}^{-2\xi_0^2}, \tag{59}$$

where we have used the expression (57) and keep only the main term with $i = n'-n$. The probability of transition between the states with $n' < n$ follows from (59) after the substitution $n \leftrightarrow n'$ and it can be obtained from (57) keeping the main term with $i = 0$.

According to (59) when the difference $\xi_0 = m_\sigma(k' - k)$ grows the transition probability falls almost exponentially. For the $\phi$ quanta with the masses $m_\sigma \sim 1$ the transitions between the states of a condensate with the close numbers $k$ and $k'$ in different potential wells $U(a)$ have the highest probability. For the mass $m_\sigma \sim 10^{-19}$ ($\sim 1\,\mathrm{GeV}$) the parameter $\xi_0 \sim 1$ for the difference $(k' - k) \sim 10^{19}$.

Nonzero transition probability points to a possibility in principle for the universe to evolve as a result of transitions between quantum states. An increase (decrease) in the mass of a condensate means an increase (decrease) in the mean value of the scale factor of the universe. From the point of view of semi-classical theory the universe will expand (contract).

Using Eq. (58) one can calculate the probability of transition of the universe from the ground (vacuum) state, $n = 0$, to any other state. It obeys the Poisson distribution

$$w(0, k \to n', k') = \frac{\langle n' \rangle^{n'}}{n'!}\, \mathrm{e}^{-\langle n' \rangle} \tag{60}$$

with the mean value $\langle n' \rangle = \xi_0^2/2$ of the quantum number $n'$. If $\langle n' \rangle \ll 1$, then this probability is small and it decreases rapidly with an increase in $n'$. The highest probability in this case has the transition $0, k \to 1, k'$,

$$w(0, k \to 1, k') = \langle n' \rangle - O(\langle n' \rangle^2). \tag{61}$$

The transition vacuum $\to$ vacuum from different potential wells $U(a)$ is given by an exponent

$$w(0, k \to 0, k') = \mathrm{e}^{-\langle n' \rangle}. \tag{62}$$

It means that the transitions from vacuum to non-vacuum states occur with the overwhelming probability. The total probability of such transitions equals to

$$w(\mathrm{vac} \to \mathrm{nonvac}) \equiv \sum_{n' \neq 0} w(0, k \to n', k') = 1 - \mathrm{e}^{-\langle n' \rangle}. \tag{63}$$

The ratio of the total transition probability (56) at $n = 0$ to probabilities of transitions between the vacuum states is equal to

$$\frac{\sum_{n'} w(0, k \to n', k')}{w(0, k \to 0, k')} = \mathrm{e}^{\langle n' \rangle}, \tag{64}$$

i.e. against a background of vacuum-vacuum transitions the probabilities of transitions into non-vacuum states look like exponentially high.

## 6. Conclusion

In this paper we calculate the spectrum of stationary states (39) and the wave functions (36), (38) of the homogeneous and isotropic universe in the epoch of matter-energy production from a primordial uniform scalar field on basis of the

exact solution of Eq. (19) of quantum model. Produced matter-energy represents itself a condensate of excitation quanta of oscillations of a scalar field above its true vacuum state. The mass of a condensate (35) and the mean value of the scale factor (43) in a given state of the universe are connected between themselves by the linear expression (44). Let us note that the condition (46) is a mathematical formulation of the Mach's principle proposed by Sciama [19] (see also [20]). The universe in an arbitrary state (36) is described by the superposition of the states with all possible masses of a condensate. The nucleation rate of the universe from the initial cosmological singularity point (51) appears to be non-zero, while the total nucleation rate obeys the exponential (explosive) law (52). The nucleation of the universe takes place as a result of its transition from the initial cosmological singularity point with the extremely rigid equation of state of a primordial scalar field into the state with the non-zero mass of a condensate with the vacuum equation of state. The universe being nucleated in the ground (vacuum) state has the Planck parameters. The evolution of the universe is described as transitions with the non-zero probability (59) between the states of the universe with different masses of a condensate. An increase (decrease) in this mass leads to an expansion (contraction) of the universe.

The classical universe can be imagined as a ball which is situated in the minimum of the effective potential $U(a)$ (40) and moves together with this minimum in the direction of large $a$ with grows of its mass. From the point of view of classical theory such a motion (expansion of the universe) can continue for an arbitrary long period of time. System has no restrictions in dimensions and mass.

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